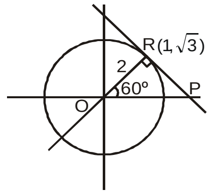
TANGENT & NORMAL

EXERCISE – I   HINTS & SOLUTIONS

Sol.1 **B**



$\tan 60º = \frac{RP}{2}$

$RP = 2\sqrt{3}$

Area of $\Delta OPR = \frac{1}{2} \times (OR) \times (RP)$

$= \frac{1}{2} \times 2 \times 2\sqrt{3}$

$= 2\sqrt{3}$ sq. units

Sol.2 **A**

$y = -\sqrt{x} + 2$

$y = \tan(\tan^{-1} x) = x$

$x = -\sqrt{x} + 2$ For intersection point

$(x-2)^2 = x$

$x = 1, \quad x = 4$

$y = 1, \quad y = 4$ (4, 4) reject

$x = 1, y = 1$ P(1, 1)

$y = -\sqrt{x} + 2$

$\frac{dy}{dx} = -\left.\frac{1}{2\sqrt{x}}\right|_p = -\frac{1}{2}$ = slope of the tangent

slope of the normal = 2

$y - 1 = 2(x - 2)$

$2x - y - 1 = 0$

Sol.3 **B**

Given that $M_N = -1$

$M_T = 1$

$ay^2 = x^3$

$2ay\frac{dy}{dx} = 3x^2 \Rightarrow \frac{dy}{dx} = \left.\frac{3x^2}{2ay}\right|_p = \frac{3x_1^2}{2ay_1}$

$\frac{3x_1^2}{2ay_1} = 1$ at point P

$3x_1^2 = 2ay_1$ $\quad ay_1^2 = x_1^3$

$9x_1^4 = 4a^2 y_1^2$

$9x_1^4 = 4a^2\left(\frac{x_1^3}{a}\right)$

$x_1 = \frac{4a}{9}$

Sol.4 **D**

$x = a(\theta + \sin\theta)$ $\quad y = a(1 + \cos\theta)$

$\frac{dx}{d\theta} = a(1 + \cos\theta)$ $\quad \frac{dy}{d\theta} = -a\sin\theta$

$\frac{dy}{dx} = -\frac{a\sin\theta}{a(1+\cos\theta)} = -\left.\frac{\sin\theta}{1+\cos\theta}\right|_{\theta=\frac{\pi}{3}}$

$= \frac{-\sqrt{3}/2}{1+1/2} = -\frac{1}{\sqrt{3}}$

$\frac{dy}{dx} = -\frac{1}{\sqrt{3}} = \tan\alpha$

$\alpha = \frac{5\pi}{6}$

Sol.5 **C**

$\frac{a}{x^2} + \frac{y}{y^2} = 1$ $\quad P(x_1, y_1)$

$-\frac{2a}{x^3} - \frac{2b}{y^3}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\left.\frac{ay^3}{bx^3}\right|_p = -\frac{ay_1^3}{bx_1^3}$

$y - y_1 = -\frac{ay_1^3}{bx_1^3}(x - x_1)$

put y = 0

$\frac{x_1^3 b}{ay_1^2} + x_1 = x$

$x = x_1 + \frac{x_1^3}{a}\left(1 - \frac{a}{x_1^2}\right)$

$= x_1 + \frac{x_1^3}{a} - x_1$

$x = \frac{x_1^3}{a}$

Sol.6 B

$y = 1 - ax^2$ $\qquad y = x^2$

$P(x_1, y_1)$

$\frac{dy}{dx} = -2ax\Big|_p$; $\frac{dy}{dx} = 2x\Big|_p$ Point of intasection

$m_1 = -2ax_1$ $\qquad m_2 = 2x_1$

$m_1 \times m_2 = -1$ $\qquad y_1 = 1 - ax_1^2$

$-2ax_1 \times 2x_1 = -1$ $\qquad y_1 = x_1^2$

$4ax_1^2 = 1$ $\qquad 1 - ax_1^2 = x_1^2$

$\frac{4a}{1+a} = 1$ $\qquad x_1^2 = \frac{1}{1+a}$

$a = \frac{1}{3}$

Sol.7 A

$y^2 = 8x$ let the point $P(2t^2, 4t)$

$2y\,y_1' = 8$

$y' = \frac{4}{y}\Big|_p = \frac{4}{4t} = \frac{1}{t}$

Normal

$y - 4t = -t(x - 2t^2)$

this normal will pass through the centre of given circle $(0, -6)$

$-6 - 4t = -t(0 - 2t^2)$

$2t^3 + 4t + 6 = 0$

$t^3 + 2t + 3 = 0$

$t = -1$

$P(2, -4)$

Sol.8 A

$\sqrt{x} + \sqrt{y} = 3$ $\qquad P(4, 1)$

$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\sqrt{\frac{y}{x}}\Big|_p = -\frac{1}{2}$

$L_{ST} = \left|\frac{y_1}{m}\right|$

$= \left|\frac{1}{-1/2}\right| = 2$

Sol.9 A

$\lambda = \frac{(\text{Length of normal})^2}{(\text{Length of tangent})^2} = \frac{(y_1\sqrt{1+m^2})^2}{\left(\frac{y_1\sqrt{1+m^2}}{m}\right)^2}$

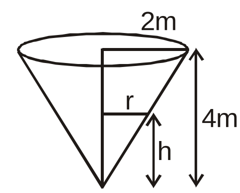
$= m^2 \quad = \frac{(y_1 m)}{(y1/m)} = \frac{L_{SN}}{L_{ST}}$

Sol.10 B

By Triangle

$\frac{r}{h} = \frac{2}{4} \Rightarrow r = \frac{h}{2}$

$v = \frac{1}{3}\pi r^2 h$



$= \frac{\pi}{3}\left(\frac{h}{2}\right)^2 .h$

$v = \frac{\pi}{12} h^3$ $\qquad \frac{dv}{dt} = 77000$ cm^3/mis.

$\frac{dv}{dt} = \frac{\pi}{4} h^2 . \frac{dh}{dt}$

$\frac{77000 \times 4}{\frac{22}{7} \times 70 \times 70} = \frac{dh}{dt}$

$\frac{dh}{dt} = 20$ cm/min.

Sol.11 B

$y = \frac{2}{3}x^3 - 2ax^2 + 2x + 5$

$\frac{dy}{dx} = 2x^2 - 4ax + 2$

makes an acute angle that means $\frac{dy}{dx}$ always positive or equals to zero $\frac{dy}{dx} \geq 0$

$2x^2 - 4ax + 2 \geq 0$

$D \leq 0$

$16a^2 - 4(2)(2) \leq 0$

$a^2 - 1 \leq 0$

$-1 \leq a \leq 1$

Sol.12 D

$\frac{x}{a} + \frac{y}{b} = 1 \qquad y = be^{-x/a}$

$\frac{dy}{dx} = -\frac{b}{a}e^{-x/a}$

$\frac{y}{b} = -\frac{y}{b} + 1 \qquad -\frac{b}{a}e^{-x/a} = -\frac{b}{a}$

$y = -\frac{b}{a}x + b \qquad e^{-x/a} = 1$

$x_1 = 0$
$y_1 = b$

Point (0, b)

Sol.13 B

$y^2 = 4a(x + a\sin\frac{x}{a})$

Let point P (h, k)

$2y\, y' = 4a\left(1+\cos\frac{x}{a}\right)$

slope should be equal to zero

$\cos\frac{x}{a} = -1$

point will lie on the curves also

$k^2 = 4a(h + a\sin\frac{h}{a}) \qquad \sin\frac{h}{a} = 0$

$k^2 = 4ah \qquad \Rightarrow \qquad y^2 = 4ax$

Sol.14 D

$x = \sec^2 t \qquad y = \cot t$

$\frac{dx}{dt} = 2\sec^2 t\tan t \qquad \frac{dy}{dt} = -\operatorname{cosec}^2 t$

$\frac{dy}{dx} = -\frac{\operatorname{cosec}^2 t}{2\sec^2 t\tan t}$

$= -\frac{\cot^2 t}{2\tan t} = -\frac{1}{2\tan^3 t}$

$\frac{dy}{dx} = -\frac{1}{2} \qquad P(2, 1)$

at $t = \frac{\pi}{4} \qquad x_1 = \sec^2\frac{\pi}{4} = 2$

$y_1 = \cot\frac{\pi}{4} = 1$

$y - 1 = -\frac{1}{2}(x - 2)$

$2y - 2 = -x + 2$

$2y + x = 4$

curve

$x = 1 + \tan^2 t$

$x = 1 + \frac{1}{y^2}$

Let $Q(x_1, y_1)$

Solve tangent with curve equation

$4 - 2y = 1 + \frac{1}{y^2}$

$y = 1, -\frac{1}{2}$

$y = -\frac{1}{2} \qquad \Rightarrow \qquad x = 5$

$Q(5, -\frac{1}{2})$

$P(2, 1)$

$PQ = \frac{3}{2}\sqrt{5}$

Sol.15 B

$y = a^{1-n}x^n$

$\frac{dy}{dx} = n\, a^{1-n}x^{n-1}$

$L_{SN} = |n\, a^{1-n}x_1^{n-1}y_1|$
$= |n\, a^{1-n}x_1^{n-1}a^{1-n}x_1^n|$
$= |n\, a^{2-2n}x_1^{2n-1}|$

$2n - 1 = 0$

$n = 1/2$

Sol.16 B

$x^3 + pxy^2 = -2 \quad ; \quad 3x^2y - y^3 = 2$

$3x^2 + P(y^2 + 2xyy') = 0 \ ; \ 6xy + 3x^2y' - 3y^2y' = 0$

$m_1 = y' = \frac{3x^2 + py^2}{-2pxy} \qquad m_2 = y' = -\frac{2xy}{3x^2 - 3y^2}$

$m_1 \times m_2 = -1$

$\frac{(3x^2 + py^2)}{-2pxy} \times \frac{(-6xy)}{(3x^2 - 3y^2)} = -1$

$\frac{3}{p}\frac{(3x^2 + py^2)}{(3x^2 - 3y^2)} = -1$

p = –3 only possible

Sol.17 B

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\frac{2x}{a^2} - \frac{2yy'}{b^2} = 0$

$xy = c^2$

$y + xy' = 0$

$m_2 = y' = -\frac{y}{x}$

$m_1 = y' = \frac{b^2}{a^2}\frac{x}{y}$

$m_1 \times m_2 = -1$

$\frac{b^2}{a^2}\frac{x_1}{y_1} \times \left(-\frac{y_1}{x_1}\right) = -1$

$b^2 = a^2$

Sol.18 B

$y = \frac{a}{2}\,(e^{x/a} + e^{-x/a})$ $\qquad$ $P(x_1, y_1)$

$$\frac{dy}{dx} = \frac{a}{2}\left(\frac{1}{a}e^{x/a} - \frac{1}{a}e^{-x/a}\right)\Bigg|_p$$

$$\frac{dy}{dx} = \frac{1}{2}\left(e^{x_1/a} - e^{-x_1/a}\right)$$

$L_N = y_1\,\sqrt{1+m^2}$

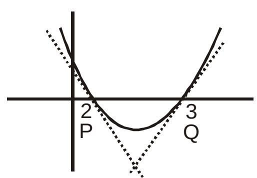
$$= \frac{a}{2}\,(e^{x_1 a} + e^{-x_1/a})\,\sqrt{1+\frac{1}{4}(e^{x_1/a} - e^{-x_1/a})^2}$$

$$a \times L_N = \frac{a}{2}\,(e^{x_1/a} - e^{-x_1/a})\ \ \frac{a}{2}\,(e^{x_1/a} + e^{-x_1/a})$$

$a \times L_N = y_1^{\,2}$
$y_1^{\,2} = a \times L_N$
quantity = a

Sol.19 A

$y = x^2 - 5x + 6$
$y = (x - 2)(x - 3)$



$$\frac{dy}{dx} = 2x - 5$$

$$m_1 = \frac{dy}{dx}\bigg|_P = -1$$

$$m_2 = \frac{dy}{dx}\bigg|_Q = 1$$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{-1-1}{1-1}\right| = \infty$$

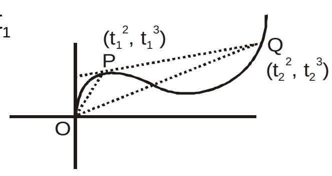
$$\theta = \frac{\pi}{2}$$

Sol.20 B

$y^2 = x^3$
Let $P(t_1^{\,2}, t_1^{\,3})$
$2y\,y' = 3x^2$

$$y' = \frac{3x^2}{2y}$$

$$y'|_P = \frac{3\,t_1^4}{2\,t_1^3} = \frac{3}{2}\,t_1$$



$m_{OP} = \tan\alpha = t_1$
$m_{OQ} = \tan\beta = t_2$

$$\frac{\tan\alpha}{\tan\beta} = \frac{t_1}{t_2}$$

$$M_{PQ} = \frac{t_2^3 - t_1^3}{t_2^2 - t_1^2} = \frac{(t_2 - t_1)(t_2^{\,2} + t_1 t_2 + t_1^{\,2})}{(t_2 - t_1)(t_2 + t_1)}$$

M_{PQ} = Slope of tangent at point p

$M_{PQ} = y'|_p$

$$\frac{t_2^{\,2} + t_1 t_2 + t_1^{\,2}}{(t_2 + t_1)} = \frac{3}{2}\,t_1$$

$t_1 = -2t_2$

$$\frac{t_1}{t_2} = -2$$

$$\frac{\tan\alpha}{\tan\beta} = -2$$

Sol.21 A

$V = \pi r^2 h$

$$\frac{dv}{dt} = \pi r^2\,\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{dv/dt}{\pi r^2} = \frac{1}{9\pi}\ \text{m/min.}$$

Sol.22 B

$x^2 + y^2 - 2x - 3 = 0$

$$2x + 2y\,\frac{dy}{dx} - 2 = 0$$

$$\frac{dy}{dx} = \frac{2-2x}{2y}\bigg|_p = \frac{2-2x_1}{2y_1}\bigg|_p$$

$2 - 2x_1 = 0$
$x_1 = 1$
$1 + y_1^{\,2} - 2 - 3 = 0$
$y_1^{\,2} = 4$
$y_1 = \pm 2$
Two points (1, 2) and (1, –2)

Sol.23 B

Subtangent = Subnormal
$L_{ST} = L_{SN}$

$$\frac{y_1}{m} = y_1 m$$

$m^2 = 1$

$$L_T = \left|\frac{y_1\sqrt{1+m^2}}{m}\right| = y_1\sqrt{2}$$

$= \sqrt{2}\,y_1 = \sqrt{2}$ ordinate

Sol.24 B

$x = a(\theta + \sin\theta)$ $\quad y = a(1 - \cos\theta)$

$$\frac{dy}{dx} = \frac{a\sin\theta}{a(1+\cos\theta)}\bigg|_{\theta=\frac{x}{2}} = 1 \quad\Rightarrow\quad m = 1$$

$x_1 = a\left(\frac{x}{2} + 1\right)$ $\qquad\qquad$ $y_1 = a$

$L_N = y_1\sqrt{1+m^2}$
$L_N = \sqrt{2}\,a$

Sol.25 B

$3x + 4y = c$ $\qquad\qquad$ $\frac{x^4}{2} = x + y$

$y = -\frac{3}{4}x + \frac{c}{4}$ $\qquad\qquad$ $\frac{4x^3}{2} = 1 + y'$

$c = 3x_1 + 4y_1$ $\qquad\qquad$ $y' = 2x^3 - 1$

$= \frac{3}{2} - \frac{60}{32}$ $\qquad\qquad$ $2x_1^3 - 1 = -\frac{3}{4}$

$2x_1^3 = \frac{1}{4}$

$c = -\frac{12}{32}$ $\qquad\qquad$ $x_1 = \frac{1}{2}$

unique value of c

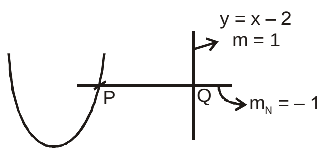
$$y_1 = \frac{x_1^4}{2} - x_1$$

$$= \frac{1}{3^2} - \frac{1}{2} = -\frac{15}{32}$$

Sol.26 B

$y = x^2$

$\frac{dy}{dx} = 2x\Big|_p = 2x_1$



$2x_1 = 1$

$x_1 = \frac{1}{2}$ $\quad P\left(\frac{1}{2}, \frac{1}{4}\right)$

$y_1 = \frac{1}{4}$

equation of normal

$y - \frac{1}{4} = -\left(x - \frac{1}{2}\right)$

$4y - 1 = -4x + 2$

$4y + 4x = 3$ ....(1) $\quad$ Intersection point

$y = x - 2$ ....(2)

$Q\left(\frac{11}{8}, -\frac{5}{8}\right)$

Sol.27 B

$x^2y = 1 - y$

$xy = 1 - y$

$x^2y = xy$

$x^2y - xy = 0$

$xy(x - 1) = 0$

$xy = 0$ & $x = 1$ $\qquad \Rightarrow y = 1/2$

$x = 0 \quad \Rightarrow y = 1$ $\qquad$ POI $\quad P(1, 1/2)$

$y = 0 \quad 0 = 1$ not possible $\;Q(0, 1)$

$x^2y = 1 - y$

$2xy + x^2y' = -y'$

$y' = \frac{-2xy}{1 + x^2}\Big|_p = \frac{-2(1)(1/2)}{1+1} = -\frac{1}{2}$

$y'|_Q = 0$

at P tangent $\;y - \frac{1}{2} = -\frac{1}{2}(x - 1)$ ....(1)

at Q tangent $\;y - 1 = 0 \Rightarrow y = 1$ $\qquad$ ....(2)

Intarsection of (1) & (2)

$\frac{1}{2} = -\frac{1}{2}(x - 1)$

$x - 1 = -1$

$x = 0$

POI $(0, 1)$

Sol.28 A

$x^ny = a^n$

$nx^{n-1}y + x^n y^1 = 0$

$$y' = -\frac{x^{n-1}y}{x^n}$$

$$y' = -\frac{y}{x}$$

Equation of tangent

$$y - y_1 = -\frac{y_1}{x_1}(x - x_1)$$

at x-axis $\qquad y = 0 \quad x_1 = x - x_1 \;\Rightarrow\; x = 2x_1$

at y-axis $\qquad m = 0 \quad y = 2y_1$

Area of $\Delta = \frac{1}{2}(2x_1)(2y_1)$

$= 2x_1y_1$

$= 2x_1 \frac{a^n}{x_1^n}$

$= 2a^n x_1^{(1-n)}$

$1 - n = 0 \qquad\qquad \Rightarrow n = 1$

Sol.29 B

$y^2 = x^3 + x^2$

curve passes through origin (0, 0)

If we want to drawn the tangent at (0, 0)

$y^2 = x^2$

$y = x$ $\quad$ and $\quad$ $y = -x$

Sol.30 A

$x = t^2 - 1$ $\qquad$; $\;y = t^2 - t$

$\frac{dx}{dt} = 2t$ $\qquad$ $\frac{dy}{dt} = 2t - 1$

$\frac{dy}{dx} = \frac{2t-1}{2t}$ $\qquad$ $\frac{dy}{dx} \to \infty$ $\;$ or $\;$ $\frac{dx}{dy} = 0$

$\frac{2t}{2t-1} = 0$

$t = 0$